So far, we've seen many examples of how to write a function $f(x)$ as a series $\sum_{n=0}^{\infty} c_n x^n$. In the examples we done, there has always been a connection to the series for $\frac{1}{1-x}$. What if there was no connection?

Let's begin a different way; let's assume $f(x) = \sum_{n=0}^{\infty} c_n x^n$ or $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ and discover what the c_n's must be.

Although these two series are pretty much the same, they go by different names (to honor different dead Brits who discovered them). The nice series with $a = 0$ are called **MacLaurin Series** while the series with $a \neq 0$ are called **Taylor Series**. To find the c_n's this we will take derivatives, which gives a nice pattern.

$$c_n = \frac{f^{(n)}(0)}{n!}$$

MacLaurin Series

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Taylor Series

Our goal, as always, is to build recipes for functions. Specifically we are going to consider three functions that have very nice coefficients c_n for their MacLaurin series. We will then use these to find Taylor series.

MacLaurin Series Recipes

	Function		Power Series	Valid "x"
4.	$e^{\text{“}x\text{”}}$	=	$\sum_{n=0}^{\infty} \frac{(\text{“}x\text{”})^n}{n!}$	$-\infty < \text{“}x\text{”} < \infty$
5.	$\cos(\text{“}x\text{”})$	=	$\sum_{n=0}^{\infty} \frac{(-1)^n (\text{“}x\text{”})^{2n}}{(2n)!}$	$-\infty < \text{“}x\text{”} < \infty$
6.	$\sin(\text{“}x\text{”})$	=	$\sum_{n=0}^{\infty} \frac{(-1)^n (\text{“}x\text{”})^{2n+1}}{(2n+1)!}$	$-\infty < \text{“}x\text{”} < \infty$

Example: Write $f(x) = e^{-x^2}$ as a power series.

Here "x" $= -x^2$, so

$$f(x) = e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

There is a way to use MacLaurin Series to find Taylor Series. This involves a little bit of algebra

Example: Find the Taylor Series for $f(x) = e^{3x}$ for $a = 2$. Set $t = x - 2$ (in general, make $t = x - a$). Then $t + 2 = x$. Substitute and do algebra:

$$e^{3x} = e^{3(t+2)} = e^{3t} \cdot e^{6} = e^{6} \cdot e^{3t} = e^{2} \cdot \sum_{n=0}^{\infty} \frac{(3t)^n}{n!} \qquad \text{“}x\text{”} = 3t$$

$$= e^{6} \cdot \sum_{n=0}^{\infty} \frac{3^n t^n}{n!} = e^{6} \cdot \sum_{n=0}^{\infty} \frac{3^n (x-2)^n}{n!} \qquad \text{since } t = x - 2$$

$$= \sum_{n=0}^{\infty} \frac{e^{6} 3^n (x-2)^n}{n!}$$

Essentially, we “re-centered” the function to have center 0, used our MacLaurin Series (which always has center 0), and then shifted back using algebra. Speaking of algebra, here are some rules that will come in handy (and be given to you on the final exam).

Algebra and Trigonometry Identities

- $e^{A+B} = e^{A} \cdot e^{B}$
- $e^{A-B} = \dfrac{e^{A}}{e^{B}}$
- $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$
- $\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$
- $\ln(A \cdot B) = \ln(A) + \ln(B)$

Answer the following questions.

1. $f(x) = x^3 \cdot \cos(2x^2)$. Find the MacLaurin Series.

2. $f(x) = \sin(\pi x)$. Find the Taylor Series at $a = \dfrac{\pi}{2}$.

3. $f(x) = \ln(2x)$. Find the Taylor Series at $a = 3$

4. Integrate using Power Series. $\displaystyle\int \sin(3x^2)\ \mathrm{d}x$